

MULTIPLE SOLUTIONS OF p -BIHARMONIC EQUATIONS WITH NAVIER BOUNDARY CONDITIONS

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ABSTRACT. In this paper, exploiting variational methods, the existence of multiple weak solutions for a class of elliptic Navier boundary problems involving the p -biharmonic operator is investigated. Moreover, a concrete example of an application is presented.

1. INTRODUCTION

Motivated also by the fact that such kind of problems are used to describe a large class of physical phenomena, many authors looked for multiple solutions of elliptic equations involving biharmonic and p -biharmonic type operators: see, for instance, the papers [6, 10, 18, 19, 20]. In the present work we are interested in the existence of multiple weak solutions for the following nonlinear elliptic Navier boundary value problem involving the p -biharmonic operator

$$(H_\lambda^f) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(x, u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open bounded subset of \mathbb{R}^N with a smooth enough boundary $\partial\Omega$, $p > \max\{1, N/2\}$, Δ is the usual Laplace operator, λ is a positive parameter and f is a suitable continuous function defined on the set $\bar{\Omega} \times \mathbb{R}$.

For $p = 2$, the linear operator $\Delta^2 u := \Delta(\Delta u)$ is the iterated Laplace which multiplied with a positive constant often occurs in Navier-Stokes equations as a viscosity coefficient. Moreover, its reciprocal operator denoted $(\Delta^2 u)^{-1}$ is the celebrated Green operator (see [12]).

In [19], a Navier boundary value problem is treated where the left-hand side of the equation involves an operator that is more general than the p -biharmonic. Meanwhile in [14], a concrete example of application of such mathematical model to describe a physical phenomena is also pointed out.

Further, by using the abstract and technical approach developed in [2, 3, 5], the authors are interested in looking for the existence of infinitely many weak solutions of perturbed p -biharmonic equations.

Here, requiring a suitable growth of the primitive of f , we are able to establish suitable intervals of values of the parameter λ for which the problem (H_λ^f) admits at least three weak solutions.

More precisely, the main result ensures the existence of two real intervals of parameters Λ_1 and Λ_2 such that, for each $\lambda \in \Lambda_1 \cup \Lambda_2$, the problem (H_λ^f) admits

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at least three weak solutions whose norms are uniformly bounded with respect to every $\lambda \in \Lambda_2$ (see Theorem 3.1).

Our method is mostly based on a useful critical point theorem given in [1, Theorem 3.1] (see Theorem 2.1 below). We also cite a recent monograph by Kristály, Rădulescu and Varga [9] as a general reference on variational methods adopted here.

The obtained results are related to some recent contributions from [10, Theorem 1] where, by using a critical point result from [16], the existence of at least three weak solutions has been obtained (see also [11, Theorem 1]). We emphasize that, in our cases, on the contrary of the above mentioned works, we give a qualitative analysis of the real intervals Λ_i ($i = 1, 2$) for which problem (H_λ^f) admits multiple weak solutions (see, for details, Remarks 2.2 and 3.3).

As an example, we present a special case of our results (see Theorem 3.4 and Remark 3.5 for more details) on the existence of two nontrivial weak solutions.

Theorem 1.1. *Let $p > \max\{1, N/2\}$ and $f : \mathbb{R} \rightarrow [0, +\infty[$ be a continuous and nonzero function. Hence, consider the following autonomous problem*

$$(G_\lambda^f) \quad \begin{cases} \Delta(|\Delta u|^{p-2}\Delta u) = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that there exists a real constant $\gamma > 0$ such that $f(t) = 0$ for every $t \in [-\gamma, \gamma]$, in addition to

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{s-1}} = 0,$$

for some $1 \leq s \leq p$.

Then there exist two real intervals of parameters Λ'_1 and Λ'_2 such that: for every $\lambda \in \Lambda'_1$ problem (G_λ^f) admits two distinct nontrivial weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and, moreover, for each $\lambda \in \Lambda'_2$ there are two distinct nontrivial weak solutions in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ uniformly bounded in norm with respect to the parameter λ .

For completeness, we refer the reader interested in fourth-order two-point boundary value problems to papers [7, 8, 13, 15] and references therein.

The plan of the paper is as follows. Section 2 is devoted to our abstract framework, while Section 3 is dedicated to the main results and their consequences in the autonomous case. A concrete example of an application is then presented (see Example 3.6).

2. PRELIMINARIES

Here, and in the sequel, Ω is an open bounded subset of \mathbb{R}^N , $p > \max\{1, N/2\}$, while X denotes a separable and reflexive real Banach space $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ endowed with the norm

$$(1) \quad \|u\| = \left(\int_{\Omega} |\Delta u(x)|^p dx \right)^{1/p}, \quad \forall u \in X.$$

The Rellich-Kondrachov theorem assures that X is compactly imbedded in $C^0(\bar{\Omega})$, whenever

$$(2) \quad k := \sup_{u \in X \setminus \{0\}} \frac{\|u\|_{C^0(\bar{\Omega})}}{\|u\|} < +\infty,$$

where $\|u\|_{C^0(\bar{\Omega})} := \sup_{x \in \bar{\Omega}} |u(x)|$, for every $u \in X$.

Moreover, if $N \geq 3$, $\partial\Omega$ is of class $C^{1,1}$ and $p \in]N/2, +\infty[$, due to Theorem 2 and [17, Remark 1], one has the following upper bound

$$k \leq \text{meas}(\Omega)^{\frac{2}{N} + \frac{1}{p'} - 1} \frac{\Gamma(1 + N/2)^{2/N}}{N(N-2)\pi} \left[\frac{\Gamma(1 + p')\Gamma(N/(N-2) - p')}{\Gamma(N/(N-2))} \right]^{1/p'},$$

where Γ is the Gamma function, p' the conjugate exponent of p and “ $\text{meas}(\Omega)$ ” denotes the Lebesgue measure of Ω .

For our aim, the main tool is a critical points theorem contained in [1, Theorem 3.1] which we recall here for the reader's convenience.

Theorem 2.1. *Let X be a separable and reflexive real Banach space; $\Phi : X \rightarrow \mathbb{R}$ a nonnegative, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and $\Psi : X \rightarrow \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that there exists $u_0 \in X$ such that*

$$\Phi(u_0) = \Psi(u_0) = 0,$$

and that

$$(i) \quad \lim_{\|u\| \rightarrow \infty} (\Phi(u) - \lambda\Psi(u)) = +\infty,$$

for all $\lambda \in [0, +\infty[$. Further, assume that there are $r > 0$ and $\bar{u} \in X$ such that:

$$(ii) \quad r < \Phi(\bar{u});$$

$$(iii) \quad \sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u) < \frac{r}{r + \Phi(\bar{u})} \Psi(\bar{u}).$$

Then, for each

$$\lambda \in \Lambda_1 := \left[\frac{\Phi(\bar{u})}{\Psi(\bar{x}) - \sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u)}, \frac{r}{\sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u)} \right],$$

the equation

$$(3) \quad \Phi'(u) - \lambda\Psi'(u) = 0,$$

has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subset \left[0, \frac{hr}{r \frac{\Psi(\bar{u})}{\Phi(\bar{u})} - \sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u)} \right],$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, the equation (8) has at least three solutions in X whose norms are less than σ .

Note that, in the above result, the symbol $\overline{\Phi^{-1}([-\infty, r])}^w$ denotes the weak closure of the sublevel $\Phi^{-1}([-\infty, r])$. For completeness, given an operator $S : X \rightarrow X^*$, we say that S admits a continuous inverse on X^* if there exists a continuous operator $T : X^* \rightarrow X$ such that $T(S(x)) = x$ for all $x \in X$.

Remark 2.2. As observed in [1, Remark 2.1], the real intervals Λ_1 and Λ_2 in Theorem 2.1 are such that either

$$\Lambda_1 \cap \Lambda_2 = \emptyset,$$

or

$$\Lambda_1 \cap \Lambda_2 \neq \emptyset.$$

In the first case, we actually obtain two distinct open intervals of positive real parameters for which equation (8) admits two nontrivial solutions; otherwise, we achieve only one interval of positive real parameters, precisely $\Lambda_1 \cup \Lambda_2$, for which equation (8) admits three solutions and, in addition, the subinterval Λ_2 for which the solutions are uniformly bounded.

3. MAIN RESULTS

Let

$$\tau := \sup_{x \in \Omega} \text{dist}(x, \partial\Omega).$$

Simple calculations show that there is $x_0 \in \Omega$ such that $B(x_0, \tau) \subseteq \Omega$, where $B(x_0, \tau)$ denotes the open ball with center x_0 and radius τ . Now, fix $\delta > 0$ and consider the function $u_\delta \in X$ defined by

$$u_\delta(x) := \begin{cases} 0 & \text{if } x \in \bar{\Omega} \setminus B(x^0, \tau) \\ 16 \frac{l^2}{\tau^4} (\tau - l)^2 \delta & \text{if } x \in B(x^0, \tau) \setminus B(x^0, \tau/2) \\ \delta & \text{if } x \in B(x^0, \tau/2), \end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$.

At this point, let

$$F(x, \xi) := \int_0^\xi f(x, t) dt, \quad \forall (x, \xi) \in \bar{\Omega} \times \mathbb{R},$$

and put

$$R_F(\tau, \delta) := \int_{B(x^0, \tau) \setminus B(x^0, \tau/2)} F(x, u_\delta(x)) \, dx.$$

Moreover, set

$$\sigma_{p,N}(\tau) := \int_{\tau/2}^\tau |2(N+2)s^2 - 3(N+1)\tau s + N\tau^2|^p s^{N-1} ds.$$

Finally, let us denote

$$K_{p,N}(\tau) := \frac{\tau^{4p} \Gamma(N/2)}{2^{5p+1} \pi^{N/2} k^p \sigma_{p,N}(\tau)},$$

and, for $\gamma > 0$, define

$$\eta(\gamma, \delta) := \frac{\tau^{4p} \Gamma(N/2) \gamma^p}{\tau^{4p} \Gamma(N/2) \gamma^p + k^p 2^{5p+1} \pi^{N/2} \delta^p \sigma_{p,N}(\tau)}.$$

With the above notations, the main result reads as follows.

Theorem 3.1. *Let $f \in C^0(\bar{\Omega} \times \mathbb{R})$ and put*

$$F(x, \xi) := \int_0^\xi f(x, t) dt, \quad \forall (x, \xi) \in \bar{\Omega} \times \mathbb{R}.$$

Assume that there exist two positive constants γ and δ such that

- (h₁) $\delta > K_{p,N}(\tau)^{1/p} \gamma$;
- (h₂) *The following inequality holds*

$$\int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx < \eta(\gamma, \delta) \left(R_F(\tau, \delta) + \int_{B(x^0, \tau/2)} F(x, \delta) dx \right).$$

Further, require that

- (h₃) *There exist a function $\alpha \in L^1(\Omega)$ and a positive constant s with $s < p$ such that*

$$F(x, \xi) \leq \alpha(x)(1 + |\xi|^s),$$

for almost every $x \in \Omega$ and for every $\xi \in \mathbb{R}$.

Then, for each

$$\lambda \in \Lambda_1 :=]\lambda_1, \lambda_2[,$$

where

$$\lambda_1 := \frac{2^{5p+1} \pi^{N/2} \sigma_{p,N}(\tau) \delta^p}{\tau^{4p} \Gamma(N/2) p \left(R_F(\tau, \delta) + \int_{B(x^0, \tau/2)} F(x, \delta) dx - \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx \right)},$$

and

$$\lambda_2 := \frac{\gamma^p}{pk^p \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx},$$

problem (H_λ^f) has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2 \subset [0, \lambda_{3,h}],$$

where

$$\lambda_{3,h} := \frac{h\gamma^p / (pk^p)}{\gamma^p \left(R_F(\tau, \delta) + \int_{B(x^0, \tau/2)} F(x, \delta) dx \right) \tau^{4p} \Gamma(N/2) - \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx},$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, problem (H_λ^f) has at least three solutions in X whose norms are less than σ .

Proof. For each $u \in X$, let $\Phi, \Psi : X \rightarrow \mathbb{R}$ defined by setting

$$\Phi(u) := \frac{\|u\|^p}{p}, \quad \Psi(u) := \int_{\Omega} F(x, u(x)) dx.$$

It is easy to verify that $\Phi : X \rightarrow \mathbb{R}$ is a nonnegative, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* . Meanwhile, Ψ is continuously Gâteaux

differentiable with compact derivative and, moreover, $\Phi(u_0) = \Psi(u_0) = 0$, where u_0 is the identically zero function in X . In particular, one has

$$\Phi'(u)(v) = \int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx,$$

and

$$\Psi'(u)(v) = \int_{\Omega} f(x, u(x)) v(x) dx,$$

for every $u, v \in X$.

Now, fixing $\lambda > 0$, if we recall that a weak solution of problem (H_{λ}^f) is a function $u \in X$ such that

$$\int_{\Omega} |\Delta u(x)|^{p-2} \Delta u(x) \Delta v(x) dx = \lambda \int_{\Omega} f(x, u(x)) v(x) dx,$$

for every $v \in X$, it is obvious that our goal is to find critical points of the energy functional $J_{\lambda} := \Phi - \lambda \Psi$.

Thanks to hypothesis (h_3) and bearing in mind (2), one has

$$\int_{\Omega} F(x, u(x)) dx \leq \|\alpha\|_{L^1(\Omega)} (1 + k^s \|u\|^s).$$

Hence

$$J_{\lambda}(u) \geq \frac{\|u\|^p}{p} - \lambda \|\alpha\|_{L^1(\Omega)} (1 + k^s \|u\|^s).$$

Therefore, due to $s < p$, the following relation holds

$$\lim_{\|u\| \rightarrow \infty} J_{\lambda}(u) = +\infty,$$

for every $\lambda > 0$.

Since J_{λ} is coercive for every positive parameter λ , condition (i) is verified. Next, consider the function $u_{\delta} \in X$. Since

$$\sum_{i=1}^N \frac{\partial^2 u_{\delta}(x)}{\partial x_i^2} = 32d \left(\frac{2(N+2)l^2 - 3\tau(N+1)l + N\tau^2}{\tau^4} \right),$$

for every $x \in B(x^0, \tau) \setminus B(x^0, \tau/2)$ and

$$\sum_{i=1}^N \frac{\partial^2 u_{\delta}(x)}{\partial x_i^2} = 0, \quad \forall x \in (\bar{\Omega} \setminus B(x^0, \tau)) \cup B(x^0, \tau/2),$$

one has

$$(4) \quad \Phi(u_{\delta}) = \frac{\|u_{\delta}\|^p}{p} = \frac{2^{5p+1} \pi^{N/2} \delta^p}{\tau^{4p} \Gamma(N/2)p} \sigma_{p,N}(\tau).$$

Put

$$r := \frac{\gamma^p}{pk^p}.$$

Now, it follows from $\delta > K_{p,N}(\tau)^{1/p} \gamma$ that $\Phi(u_{\delta}) > r$. We explicitly observe that, in view of (2), one has

$$(5) \quad \Phi^{-1}([-\infty, r]) \subseteq \{u \in C^0(\bar{\Omega}) : \|u\|_{\infty} \leq \gamma\}.$$

Moreover, taking (5) into account, a direct computation ensures that

$$(6) \quad \sup_{u \in \overline{\Phi^{-1}([-\infty, r])}^w} \Psi(u) = \sup_{u \in \Phi^{-1}([-\infty, r])} \Psi(u) \leq \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) dx.$$

At this point, by definition of u_δ , we can clearly write

$$(7) \quad \int_{\Omega} F(x, u_\delta(x)) \, dx = R_F(\tau, \delta) + \int_{B(x^0, \tau/2)} F(x, \delta) \, dx.$$

By using hypothesis (h₂), from (6) and (7), we also have

$$\sup_{u \in \overline{\Phi^{-1}(\cdot) - \infty, r}]^w} \Psi(u) < \frac{r}{r + \Phi(u_\delta)} \Psi(u_\delta),$$

taking into account that

$$\frac{r}{r + \Phi(u_\delta)} = \frac{\tau^{4p} \Gamma(N/2) \gamma^p}{\tau^{4p} \Gamma(N/2) \gamma^p + k^p 2^{5p+1} \pi^{N/2} \delta^p \sigma_{p,N}} = \eta(\gamma, \delta).$$

So conditions (ii) and (iii) are verified by taking $\bar{u} := u_\delta$. Thus, we can apply Theorem 2.1 bearing in mind that

$$\frac{\Phi(u_\delta)}{\Psi(u_\delta) - \sup_{u \in \overline{\Phi^{-1}(\cdot) - \infty, r}]^w} \Psi(u)} \leq \lambda_1,$$

and

$$\frac{r}{\sup_{u \in \overline{\Phi^{-1}(\cdot) - \infty, r}]^w} \Psi(u)} \geq \lambda_2.$$

as well as

$$\frac{hr}{r \frac{\Psi(u_\delta)}{\Phi(u_\delta)} - \sup_{u \in \overline{\Phi^{-1}(\cdot) - \infty, r}]^w} \Psi(u)} \leq \lambda_{3,h}.$$

The proof is complete. \square

Remark 3.2. Assuming that

- (j₁) $F(x, \xi) \geq 0$ for every $(x, \xi) \in (B(x^0, \tau) \setminus B(x^0, \tau/2)) \times [0, \delta]$;
- (j₂) For every $|\xi| \leq \gamma$ one has

$$\int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) \, dx < \eta(\gamma, \delta) \int_{B(x^0, \tau/2)} F(x, \delta) \, dx,$$

it follows that hypothesis (h₁) in Theorem 3.1 automatically hold.

Remark 3.3. We point out that hypothesis (h₂) in Theorem 3.1 can be stated in a more general form. Precisely, fix $x^0 \in \Omega$ and pick $r_1, r_2 \in \mathbb{R}$ with $r_2 > r_1 > 0$, such that $B(x^0, r_1) \subset B(x_0, r_2) \subseteq \Omega$. Moreover, set

$$\sigma_{p,N}(r_1, r_2) := \int_{r_1}^{r_2} |(N+2)s^2 - (N+1)(r_1 + r_2)s + Nr_1r_2|^p s^{N-1} ds,$$

and denote

$$K_{p,N}(r_1, r_2) := \frac{(r_2 - r_1)^{3p} (r_1 + r_2)^p \Gamma(N/2)}{2^{2p+1} 3^p \pi^{N/2} k^p \sigma_{p,N}(r_1, r_2)}.$$

At this point, let v_δ the function be defined as follows,

$$v_\delta(x) := \begin{cases} 0 & \text{if } x \in \bar{\Omega} \setminus B(x^0, r_2) \\ \frac{\delta(3(l^4 - r_2^4) - 4(r_1 + r_2)(l^3 - r_2^3) + 6r_1 r_2(l^2 - r_2^2))}{(r_2 - r_1)^3(r_1 + r_2)} & \text{if } x \in B(x^0, r_2) \setminus B(x^0, r_1) \\ \delta & \text{if } x \in B(x^0, r_1), \end{cases}$$

where $l := \sqrt{\sum_{i=1}^N (x_i - x_i^0)^2}$.

If δ and γ in Theorem 3.1 satisfy $\delta > K(r_1, r_2)^{1/p}\gamma$, instead of (h_1) , hypothesis (h_2) can be replaced by the following assumption, namely (h_2^*) :

$$\int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) \, dx < \frac{r}{r + \Phi(v_\delta)} \left(R_F(r_1, r_2, \delta) + \int_{B(x^0, r_1)} F(x, \delta) \, dx \right),$$

where

$$r := \frac{\gamma^p}{pk^p},$$

$$R_F(r_1, r_2, \delta) := \int_{B(x^0, r_2) \setminus B(x^0, r_1)} F(x, v_\delta(x)) \, dx,$$

and

$$\Phi(v_\delta) = \frac{\delta^p}{pk^p K_{p,N}(r_1, r_2)}.$$

Then for each

$$\lambda \in \Lambda_1^* := \left[\frac{\Phi(v_\delta)}{\Psi(v_\delta) - \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) \, dx}, \frac{\gamma^p}{pk^p \int_{\Omega} \max_{|\xi| \leq \gamma} F(x, \xi) \, dx} \right],$$

the equation

$$(8) \quad J_\lambda(u) = \Phi'(u) - \lambda \Psi'(u) = 0,$$

has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda_2^* \subset \left[0, \frac{hr}{r \frac{\Psi(v_\delta)}{\Phi(v_\delta)} - \sup_{x \in \Phi^{-1}([- \infty, r])^w} \Psi(x)} \right],$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda_2$, the equation (8) has at least three solutions in X whose norms are less than σ . It is clear that if $r_1 = \tau/2$ and $r_2 = \tau$, condition (h_2^*) coincides with (h_2) .

Now, for completeness, we analyze the autonomous case

$$(G_\lambda^f) \quad \begin{cases} \Delta(|\Delta u|^{p-2} \Delta u) = \lambda f(u) & \text{in } \Omega \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. With the above notations, let us define

$$G_F(\tau, \delta) := \int_{B(x^0, \tau) \setminus B(x^0, \tau/2)} F(u_\delta(x)) \, dx.$$

Finally, the symbol “ $\text{meas}(B(x^0, \tau/2))$ ” denotes the Lebesgue measure of the ball $B(x^0, \tau/2)$.

Theorem 3.4. *Let $f \in C^0(\mathbb{R})$ and put*

$$F(\xi) := \int_0^\xi f(t)dt, \quad \forall \xi \in \mathbb{R}.$$

Assume that there exist two positive constants γ and δ such that condition (h_1) hold in addition to

$$(h'_2) \quad F(\xi) < \frac{\eta(\gamma, \delta)}{\text{meas}(\Omega)} (G_F(\tau, \delta) + \text{meas}(B(x^0, \tau/2))F(\delta)), \text{ for every } |\xi| \leq \gamma.$$

Moreover, require that

$$(h'_3) \quad \text{There exist two positive constants } b \text{ and } s \text{ with } s < p \text{ such that}$$

$$F(\xi) \leq b(1 + |\xi|^s).$$

Then, for each

$$\lambda \in \Lambda'_1 :=]\lambda'_1, \lambda'_2[,$$

where

$$\lambda'_1 := \frac{2^{5p+1}\pi^{N/2}\sigma_{p,N}(\tau)\delta^p / \text{meas}(\Omega)}{\tau^{4p}\Gamma(N/2)p \left(\frac{G_F(\tau, \delta) + \text{meas}(B(x^0, \tau/2))F(\delta)}{\text{meas}(\Omega)} - \max_{|\xi| \leq \gamma} F(\xi) \right)},$$

and

$$\lambda'_2 := \frac{\gamma^p}{pk^p \text{meas}(\Omega) \max_{|\xi| \leq \gamma} F(\xi)},$$

problem (G_λ^f) has at least three distinct solutions in X and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda'_2 \subset [0, \lambda'_{3,h}],$$

where

$$\lambda'_{3,h} := \frac{h\gamma^p / (p \text{meas}(\Omega) k^p)}{\frac{\gamma^p (G_F(\tau, \delta) + \text{meas}(B(x^0, \tau/2))F(\delta)) \tau^{4p}\Gamma(N/2)}{2^{5p+1}k^p\pi^{N/2}\sigma_{p,N}(\tau) \text{meas}(\Omega)\delta^p} - \max_{|\xi| \leq \gamma} F(\xi)},$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda'_2$, problem (G_λ^f) has at least three solutions in X whose norms are less than σ .

Remark 3.5. The following two conditions

$$(j'_1) \quad G_F(\tau, \delta) \geq 0;$$

$$(j'_2) \quad \text{For every } |\xi| \leq \gamma \text{ one has}$$

$$F(\xi) < \eta(\gamma, \delta) \frac{\text{meas}(B(x^0, \tau/2))}{\text{meas}(\Omega)} F(\delta),$$

imply hypotheses (h'_1) in Theorem 3.4.

Furthermore, assumption (j'_1) is verified by requiring that $F(\xi) \geq 0$ for every $\xi \in [0, \delta]$. Moreover, if f is nonnegative, hypothesis (j'_1) automatically holds and (j'_2) attains a more simply form

$$F(\gamma) < \eta(\gamma, \delta) \frac{\text{meas}(B(x^0, \tau/2))}{\text{meas}(\Omega)} F(\delta).$$

Hence, Theorem 1.1 of Introduction is a direct consequence of the above observations. Indeed, let f be a nonnegative continuous function such that $f(t) = 0$ for every $t \in [-\gamma, \gamma]$. Bearing in mind that f is not identically zero, there exists $\delta > \gamma \max \{1, K_{p,N}(\tau)^{1/p}\}$, such that

$$0 = F(\gamma) < \eta(\gamma, \delta) \frac{\text{meas}(B(x^0, \tau/2))}{\text{meas}(\Omega)} F(\delta).$$

Finally, we observe that if

$$(h_3^*) \quad \lim_{|t| \rightarrow \infty} \frac{f(t)}{|t|^{s-1}} = 0,$$

for some $1 \leq s \leq p$, the functional J_λ is coercive. We give just some computations in the case $s = p$; analogous conclusion holds for $s \in [1, p[$. So, fix $\lambda > 0$ and pick $\varepsilon < 1/(\lambda k^p \text{meas}(\Omega))$. Now, by our assumption at infinity, there exists $c(\varepsilon) > 0$ such that

$$|f(t)| \leq \varepsilon |t|^{p-1} + c(\varepsilon), \quad \forall t \in \mathbb{R}.$$

Then the previous inequality gives

$$F(\xi) \leq \frac{\varepsilon}{p} |\xi|^p + c(\varepsilon) |\xi|, \quad \forall \xi \in \mathbb{R},$$

and, consequently, taking into account (2), one has

$$\Psi(u) \leq \left(\frac{\varepsilon k^p}{p} \|u\|^p + c(\varepsilon) k \|u\| \right) \text{meas}(\Omega), \quad \forall u \in X.$$

Since, for every $u \in X$, the following inequality holds

$$J_\lambda(u) \geq \left(\frac{1}{p} - \lambda \frac{\varepsilon k^p}{p} \text{meas}(\Omega) \right) \|u\|^p - \lambda c(\varepsilon) k \|u\| \text{meas}(\Omega),$$

the functional J_λ is coercive.

Thus, all the assumptions (with (h_3^*) instead of (h_3')) of Theorem 3.4 are verified and the conclusion follows. For completeness we also note that Theorem 1.1 of Introduction is still true without sign assumption on f on the half-line $] -\infty, \gamma[$.

At the end we exhibit a concrete application of our results.

Example 3.6. Let Ω be a nonempty bounded open subset of the Euclidean space \mathbb{R}^3 with a smooth boundary $\partial\Omega$ and define $f : \mathbb{R} \rightarrow \mathbb{R}$ as follows

$$f(t) := \begin{cases} 0 & \text{if } t < 2 \\ \sqrt{t-2} & \text{if } t \geq 2, \end{cases}$$

whose potential is given by

$$F(\xi) := \begin{cases} 0 & \text{if } \xi < 2 \\ \frac{2(\xi-2)^{3/2}}{3} & \text{if } \xi \geq 2. \end{cases}$$

Consider the following problem

$$(H_\lambda^f) \quad \begin{cases} \Delta^2 u = \lambda f(u) & \Omega \\ u = \Delta u = 0 & \partial\Omega. \end{cases}$$

Arguing as in Remark 3.5 we can observe that there exist two positive constants $\gamma = 2$ and

$$\delta > 2 \left\{ 1, K_{2,3}(\tau)^{1/p} \right\},$$

such that, taking into account Remark 3.5, all the conditions of Theorem 3.4 hold. Then, for each

$$\lambda \in \Lambda'_1 :=]\lambda_1^*, +\infty[,$$

where

$$\lambda_1^* := \frac{2^{10} \pi^{3/2} \sigma_{2,3}(\tau) \delta^2}{\tau^8 \Gamma(3/2) (G_F(\tau, \delta) + \text{meas}(B(x^0, \tau/2)) F(\delta))},$$

problem (H_λ^f) has at least three distinct (two nontrivial) solutions in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and, moreover, for each $h > 1$, there exists an open interval

$$\Lambda'_2 \subset [0, \lambda_{3,h}^*],$$

where

$$\lambda_{3,h}^* := \frac{2^{10} \pi^{3/2} \sigma_{2,3}(\tau) \delta^2 h}{\tau^8 \Gamma(3/2) (G_F(\tau, \delta) + \text{meas}(B(x^0, \tau/2)) F(\delta))} = h \lambda_1^*,$$

and a positive real number $\sigma > 0$ such that, for each $\lambda \in \Lambda'_2$, problem (H_λ^f) has at least three (two nontrivial) solutions in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ whose norms are less than σ .

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